J. Fluid Mech. (1974), *vol.* 65, *part* 1, *pp.* 33–44
Printed in Great Britain

Topological pumping of magnetic flux by three-dimensional convection

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(Received 17 September 1973)

Attention is drawn to a principal difference between the transfer of a horizontal magnetic field by turbulence and by three-dimensional cell convection.

If the motion of the conducting medium in the cells is such that the heated material ascends at the centre while descending along the sides of the cells, then the magnetic tubes of force will be carried downwards by peripheral flows. Discrete ascending flows separated from one another by descending material carry only closed magnetic field loops. Such loops do not transfer net magnetic flux. As a result, the magnetic flux becomes blocked at the base of the convective layer.

1. Introduction

A crucial point for any theory trying to explain the origin of the magnetic field of celestial bodies is the time scale, extent and manner of penetration (or decay) of the magnetic field into the bulk of the objects considered. If the conducting material is fixed, this time scale τ is determined by the diffusion of the field due to ohmic dissipation:

$$\tau = 4\pi\sigma l^2/c^2, \tag{1}$$

where σ is the conductivity of the material, l the length scale and c the velocity of light (the Gaussian system of units being used). For the earth this time is about 10^4 yr, for the sun and the stars $\gtrsim 10^{10}$ yr. The hypotheses on the primordial nature of the magnetic fields of the sun and of the stars (Cowling 1945) are based on the latter estimate.

If the material can move, the field can change as a result of either the development of hydromagnetic instabilities when the field itself initiates motion of the material (Kipper 1958), or simply the transfer of field by the material when the field is so weak that it does not affect the pattern of motion and plays the role of a passive admixture.

The latter situation is pertinent to the generation of magnetic field by material motion. Intensive motion of material in celestial bodies originates, as a rule, from thermal convection. Convection in the stars is of a turbulent nature. A large number of papers (see reviews of Parker 1970; Vainshtein & Zel'dovich 1971) are devoted to a consideration of field generation by the 'turbulent dynamo' mechanism. In these papers, one either does not consider the question of field penetration into the body or assumes that in the presence of turbulent convection the field is transferred as if it were a scalar admixture on a characteristic time scale

$$\tau_t = l^2/D_t = l^2/\kappa V_t l_t, \tag{2}$$

where D_t is the turbulent diffusion coefficient, V_t and l_t being the characteristic velocity and length scales of the turbulence. The dimensionless factor κ is of order unity. Calculations of Parker (1971) yield $\kappa = 0{\cdot}15$. When using (2) one assumes V_t and l_t to be equal to the velocity and length scale of convective motion, thus making no distinction between the transport properties of convection and turbulence.

The purpose of the present work is to draw attention to a principal difference between the transfer of magnetic field by convective motions and by random turbulence.

We deal here with three-dimensional thermal convection. Its velocity field possesses an ordered structure in that the entire convective layer is divided into cells at whose centres the material ascends while at the peripheries it descends (the direction of motion can be of either sense depending on the actual physical properties of the material, etc.; see Krishnamurti 1968). Convective motion with such structure is observed both under laboratory conditions (Bénard convection) and in nature (granulation and supergranulation on the sun).

In an appendix to the paper (by H. K. Moffatt) the phenomenon of topological pumping is analysed by the methods of 'mean field electrodynamics' under the approximation of low magnetic Reynolds number, and the main conclusion of the paper is confirmed.

2. The mechanism of flux pumping

Let three-dimensional Bénard convection take place in an infinite horizontal liquid layer. The flow in the entire layer has a cellular structure (hexagonal in the ideal case). The heated material ascends along the axis of each cell. Near the upper layer boundary it spreads towards the periphery of the cell and, on cooling, descends along the sides of the cell. On reaching the lower layer boundary, the material is heated, moves towards the cell axis and ascends again, and so on.

The velocity field of such three-dimensional convection has a topological feature essential to us. This is that regions with a downward velocity form a continuous network within which one can draw a continuous line connecting cells arbitrarily far away from one another. Regions with upward velocities do not possess this property. They are embedded in the network of descending material and are isolated one from another.

Imagine now an infinite ideally flexible filament to be placed on the top surface of the layer consisting of such cells. Near the top surface of the layer the material in each cell will spread from the centre towards the periphery. The flow of material will bend the filament, dragging it towards the sides of the cells. As a result, the entire filament will move into the region of downward velocities and eventually will be carried down, to the base of the layer. The filament as a whole cannot return to the top surface of the layer because of the discrete nature of the ascending flows.

The behaviour of this filament simulates that of an individual tube of magnetic field in a highly conducting medium. Such a tube will also be carried by convective flows to the base of the convective layer. The ascending flows cannot drag the tube as a whole back up. They can only tear from it closed loops. These loops do not carry any net magnetic flux. As a result, the original magnetic flux of the tube will be confined near the lower boundary of the layer.

Thus three-dimensional convection acts as a valve with respect to the magnetic field, producing, on the average, a vertical gradient of the horizontal magnetic field component. This may be called a magnetic flux pumping effect.

The transfer of a scalar field does not depend on the topology of convection.

3. Formulation of the problem and its solution

To check the above reasoning, we have considered a concrete case of the effect of three-dimensional convection on magnetic field.

The problem was formulated in a rather simplified form and solved numerically. The magnetic field was assumed to be weak and not to affect the velocity field. The convective cells were taken to be of rectangular (square) shape with the velocity field in them given by

$$\left.\begin{aligned} u = V_x &= -\sin\pi x(1+\tfrac{1}{2}\cos\pi y)\cos\pi z, \\ v = V_y &= -(1+\tfrac{1}{2}\cos\pi x)\sin\pi y\cos\pi z, \\ w = V_z &= [(1+\cos\pi x)(1+\cos\pi y)-1]\sin\pi z. \end{aligned}\right\} \tag{3}$$

The z axis is vertical, the x and y axes being normal to the cell sides. The cell occupies $0 \leqslant z \leqslant 1$, $-1 \leqslant x \leqslant +1$, $-1 \leqslant y \leqslant +1$.

This field satisfies the continuity equation for an incompressible fluid,

$$\nabla . \mathbf{V} = 0, \tag{4}$$

and has the desired topology, the fluid ascending at the centre and descending along the sides of the cells. The net flux of fluid through any plane z = constant is zero.

It is noteworthy that the three-dimensional velocity field given for rectangular (square) cells by Chandrasekhar (1961, chap. 2),

$$\left.\begin{aligned} u &= -\tfrac{1}{2}\sin\pi x\cos\pi y\cos\pi z, \\ v &= -\tfrac{1}{2}\cos\pi x\sin\pi y\cos\pi z, \\ w &= \cos\pi x\cos\pi y\sin\pi z, \end{aligned}\right\} \tag{5}$$

does not possess the required structure since here the regions of ascending and descending fluid alternate like the black-and-white chess-board pattern. One can draw arbitrarily long continuous lines through both ascending and descending regions. Therefore from the topological viewpoint, field (5) is two-dimensional.

The behaviour of a magnetic field **H** in the given velocity field is completely described by the following equations:

$$\nabla.\mathbf{H} = 0, \tag{6}$$

$$\partial\mathbf{H}/\partial t = \eta\nabla^2\mathbf{H} - (\mathbf{V}.\nabla)\,\mathbf{H} + (\mathbf{H}.\nabla)\,\mathbf{V}, \tag{7}$$

where $\eta = c^2/4\pi\sigma$ is the coefficient of magnetic field diffusion due to ohmic dissipation.

We sought a stationary solution of the problem ($\partial/\partial t \equiv 0$). Because of the periodicity and symmetry, computation was carried out for one quarter of the convective cell ($0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$). The total magnetic field flux was assumed to be non-zero only in the direction of the x axis:

$$\Phi_x = \int_0^1\int_0^1 H_x dy dz = 1, \tag{8}$$

$$\Phi_y = \int_{-1}^1\int_0^1 H_y dx dz = 0, \quad \Phi_z = \int_{-1}^1\int_{-1}^1 H_z dx dy = 0. \tag{9}$$

Condition (8) normalizes the magnetic field. In the absence of convection (or at $\sigma = 0$) one would have $H_x = 1$ and $H_z = H_y = 0$.

In the order for condition (8) to be met in the steady-state case, the horizontal surfaces $z = 0$ and $z = 1$ should be impermeable to the magnetic field. For this, we set $\sigma = \infty$ at $z \leqslant 0$ and $z \geqslant 1$. Then from Landau & Lifshitz (1957, chap. 6) we have that at $z = 0$ and $z = 1$

$$H_z = 0. \tag{10}$$

The conditions for the H_x and H_y components at $z = 0$ and $z = 1$ are obtained by equating to zero the electric field components tangential to the superconductor surfaces using the equation (Landau & Lifshitz 1957)

$$\nabla \wedge \mathbf{H} = \frac{4\pi}{c}\sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \wedge \mathbf{H}\right). \tag{11}$$

Noting that $w = 0$ and $H_z = 0$, this yields for $z = 0$ and $z = 1$

$$\partial H_x/\partial z = \partial H_y/\partial z = 0. \tag{12}$$

The boundary conditions for the sides of the convective cell ($x = 1$ and $y = 1$) and the symmetry planes ($x = 0$ and $y = 0$) are obtained from the conditions of periodicity and symmetry:

$$\partial H_x/\partial x = 0, \quad H_y = 0, \quad H_z = 0 \quad \text{at} \quad x = 0, 1, \tag{13}$$

$$\partial H_x/\partial y = 0, \quad H_y = 0, \quad \partial H_z/\partial y = 0 \quad \text{at} \quad y = 0, 1. \tag{14}$$

The solution to (6) and (7) with conditions (8)–(10) and (12)–(14) was sought in the form of expansions

$$H_x = \sum_0^\infty H^x_{ijk} \cos i\pi x \cos j\pi y \cos k\pi z, \tag{15}$$

$$H_y = \sum_0^\infty H^y_{ijk} \sin i\pi x \sin j\pi y \cos k\pi z, \tag{16}$$

$$H_z = \sum_0^\infty H^z_{ijk} \sin i\pi x \cos j\pi y \sin k\pi z. \tag{17}$$

i	$j \backslash k$	H^x_{ijk} 0	1	2	3	4	5	H^y_{ijk} 0	1	2	3	4	H^z_{ijk} 1	2	3
0	0	+1·000	+0·654	+0·540	+0·120	+0·080	+0·008	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0	1	+0·020	+1·051	+0·591	+0·135	+0·085	+0·007	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0	2	+0·263	+0·295	+0·167	+0·068	+0·025	+0·007	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0	3	+0·048	+0·128	+0·060	+0·022	+0·008	+0·003	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
1	0	0·000	−0·891	−0·437	−0·264	−0·061	−0·036	0·000	0·000	0·000	0·000	0·000	−0·891	−0·218	−0·088
1	1	−0·484	−0·538	−0·435	−0·270	−0·071	−0·040	−0·484	+0·590	+0·246	+0·029	+0·022	−1·129	−0·341	−0·099
1	2	−0·037	−0·261	−0·186	−0·085	−0·038	−0·012	−0·019	+0·088	+0·085	−0·008	+0·007	−0·438	−0·178	−0·023
1	3	−0·041	−0·028	−0·044	−0·027	−0·012	−0·003	−0·014	+0·036	+0·034	+0·002	+0·003	−0·137	−0·073	−0·011
1	4	+0·004	−0·013	−0·006	−0·006	−0·002	−0·001	+0·001	+0·006	+0·010	+0·003	+0·000	−0·035	−0·024	−0·006
2	0	0·000	−0·107	−0·039	+0·062	+0·023	+0·010	0·000	0·000	0·000	0·000	0·000	−0·214	−0·039	+0·041
2	1	+0·018	−0·212	−0·052	+0·087	+0·031	+0·015	+0·036	−0·030	−0·000	−0·026	−0·002	−0·395	−0·052	+0·067
2	2	−0·018	−0·097	−0·027	+0·028	+0·011	+0·007	−0·018	+0·009	+0·012	−0·018	−0·001	−0·212	−0·039	+0·030
2	3	−0·019	−0·026	−0·015	+0·003	+0·003	+0·002	−0·013	+0·006	+0·009	−0·004	−0·001	−0·070	−0·028	+0·006
3	0	0·000	−0·011	+0·010	+0·008	−0·005	−0·001	0·000	0·000	0·000	0·000	0·000	−0·032	+0·015	+0·008
3	1	−0·006	−0·008	+0·010	+0·012	−0·007	−0·002	−0·019	+0·017	−0·003	−0·003	+0·001	−0·040	+0·016	+0·012
3	2	−0·006	−0·004	−0·000	+0·004	−0·002	−0·001	−0·009	+0·003	−0·001	−0·002	+0·001	−0·019	+0·001	+0·006

TABLE 1. Amplitudes of first harmonics of magnetic field components, $R_m = 12$ ($H^z_{ij0} = 0$).

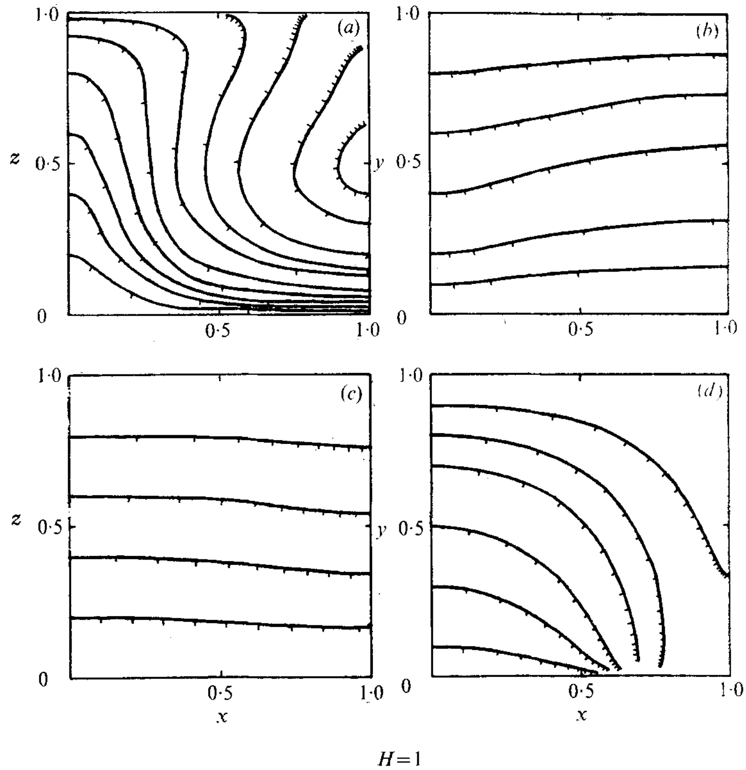

FIGURE 1. Magnetic field lines in planes with zero normal field components. The field strength is proportional to the distance between marks on curves. $R_m = 12$. (a) $y = 0$. (b) $z = 0$. (c) $y = 1$. (d) $z = 1$.

Substituting (15)–(18) into (6) and into the x and y projections of (7) yields an infinite set of linear algebraic equations for the coefficients. The set was solved numerically on a computer by iteration. For solution, the chain of equations was truncated by setting all harmonics with $i, j, k > m$ to zero. The magnitude of m was chosen so that the higher harmonics would be sufficiently small. For $R_m \leqslant 4$ it is sufficient to take $m = 4$; for $R_m \leqslant 16$, $m = 6$ (the number of algebraic equations here exceeds 1000).

4. Discussion

We carried out computations for $R_m = 4\pi\sigma l V/c^2 \leqslant 16$.

Table 1 lists the first-order coefficients for the magnetic field components for $R_m = 12$. Higher-order coefficients do not exceed 0·023 for H_x, 0·006 for H_y and 0·023 for H_z.

It is difficult to imagine and display the spatial distribution of the magnetic lines of force. Therefore figure 1 presents the lines of force only in the planes with zero normal components. Note that areas where the lines of force come closer to one another do not, generally speaking, necessarily correspond to stronger fields.

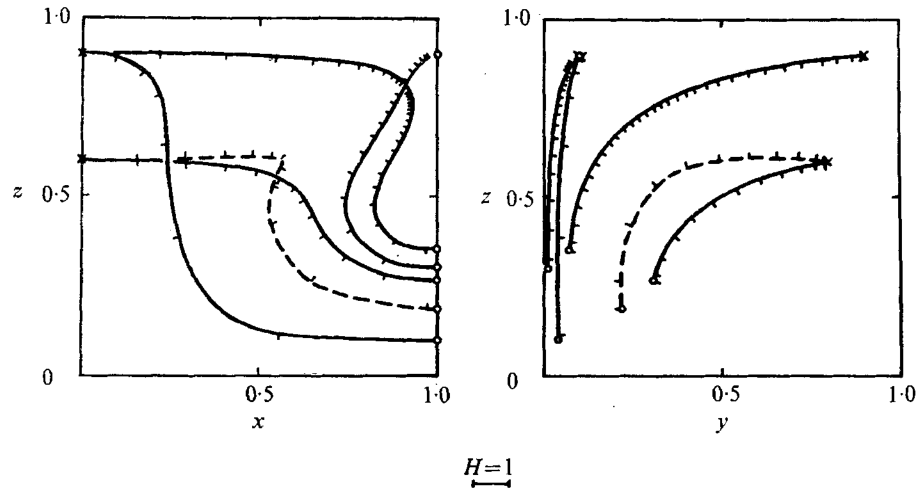

FIGURE 2. Spatial magnetic field lines. The field strength is proportional to the distance between marks on curves. ——, $R_m = 12$; - - -, $R_m = 16$. Only a few of the computed lines are shown. For the sake of simplicity, projections of the lines onto the planes $y =$ constant and $z =$ constant are given (yielding information required for the construction of the spatial curve as a whole). ×, points where the lines of force intersect the plane $x = 0$; ○, points where the lines of force intersect the plane $x = 1$.

This is caused by the three-dimensional field structure. On the figures, it is the distance between marks on lines that is proportional to the field strength.

Figure 2 provides some idea of the spatial field distribution.

The moving medium is seen to interact very strongly with the magnetic field. In the absence of motion, or at $\sigma = 0$, one would have only a uniform field $H = H_x = 1$, while $H_y = H_z = 0$. The motion of the medium results in deformation and stretching of the magnetic tubes. This will cause generation of H_y and H_z components comparable in magnitude to the original field H_x, while at $R_m \geqslant 8$ the field H_x proper will reverse the sign in a part of the volume thus even forming closed magnetic loops somewhere (figures 1 and 2).

We are interested in the behaviour of the magnetic field averaged over many cells. We have conditions (8) and (9). From (16) it follows also that

$$\int_{-1}^{1}\int_{-1}^{1} H_y dx dy = 0.$$

As for the magnetic flux pumping process discussed in § 2, its existence would produce asymmetry in the distribution of the averaged magnetic field

$$\langle H_x \rangle = \int_0^1 \int_0^1 H_x dx dy \tag{18}$$

with respect to the plane $z = \frac{1}{2}$.

As follows from (15), the magnitude of $\langle H_x \rangle$ is determined unambiguously by the coefficients H^x_{00k}, so that $\langle H_x \rangle$ will be a function of z only if these coefficients with $k \neq 0$ are non-zero. The asymmetry in question occurs if H^x_{00k} are non-zero at odd k. It turns out that these odd harmonics are zero for two-dimensional roll

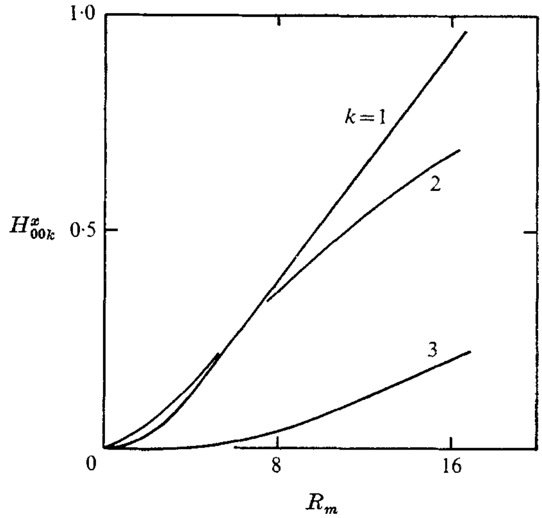

FIGURE 3. Amplitudes of first three harmonics of x field component *vs.* magnetic Reynolds number R_m.

convection and this is confirmed by the results of Weiss (1966), who also found that horizontal flux is redistributed symmetrically about the centre-plane by a layer of two-dimensional eddies.

The same holds for the three-dimensional convection with Chandrasekhar's velocity field (5). Here the three-dimensionality may be thought of as produced by deformation of rolls conserving the topological structure of the ascending and descending regions.† For motions with isolated ascending flows odd-k harmonics are non-zero.

The values of the first three coefficients H^x_{00k} for the velocity field (3) are shown in figure 3 as functions of R_m. Both the first and the second harmonics increase monotonically with growing R_m; at first ($R_m \lesssim 6$) the second one dominates over the rest. The last circumstance masks somewhat at small R_m the field asymmetry of interest to us. However the growth of the second harmonic gradually slows down while from $R_m \approx 4$ the first harmonic grows continuously in direct proportion to R_m. As for the third coefficient, it is very small and even negative at $0 < R_m \lesssim 6$, but afterwards it begins to increase.

The distribution in height of the $\langle H_x \rangle$ field averaged in the horizontal plane over many cells in the layer is presented in figure 4 for various values of R_m.

The effect of asymmetry in the net magnetic flux distribution across the convective layer is obvious. It should be stressed that the redistribution occurs despite the absence of a net material flow across the layer ($\langle w \rangle = 0$), the effect being totally due to the structural properties of the three-dimensional convection. Both the absolute and relative differences in the field strengths $\langle H_x \rangle$ between the

† Generally speaking, one can expect that any difference in the geometry or parameters (e.g. conductivity) of the ascending and descending flows will result in an asymmetric redistribution of magnetic flux.

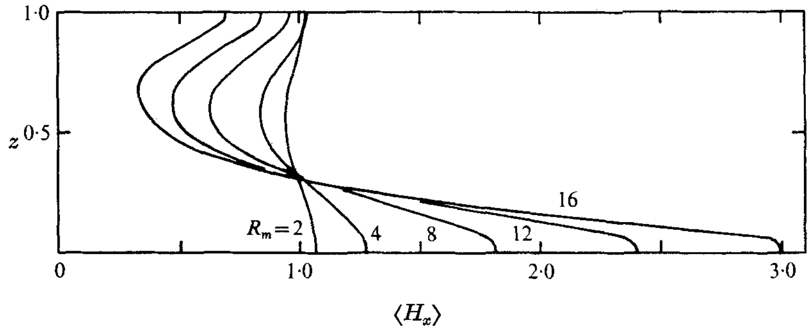

FIGURE 4. Distribution of averaged horizontal magnetic field component across convective layer.

upper ($z = 1$) and lower ($z = 0$) surfaces increase monotonically with growing R_m. Thus the original magnetic flux becomes more and more confined to the lower boundary of the convective layer.

In conclusion, it should be stressed that, in contrast to isotropic turbulence, thermal convection does not necessarily result in magnetic field dissipation. If the motion possesses a certain structure and direction which is often observed in nature, the field will be transferred into the bulk of the liquid, becoming blocked at the base of the convective layer.

Appendix

By H. K. MOFFATT, University of Cambridge

The pumping effect described in this paper may be demonstrated explicitly (without recourse to numerical methods), under the assumption that the magnetic Reynolds number is small.† The equation for the steady field $\mathbf{H}(\mathbf{x})$ is then

$$-\nabla^2\mathbf{H} = \epsilon\nabla\wedge(\mathbf{u}\wedge\mathbf{H}), \tag{A 1}$$

where $\epsilon \ll 1$. We are primarily interested in the mean field $\mathbf{H}_0(z) = \langle\mathbf{H}(\mathbf{x})\rangle$ averaged over horizontal sections of the flow; this satisfies the equation

$$-d^2\mathbf{H}_0/dz^2 = \epsilon\nabla\wedge\mathscr{E}, \tag{A 2}$$

where $\mathscr{E}(z) = \langle\mathbf{u}\wedge\mathbf{H}\rangle$ is the mean electromotive force generated. Under the boundary conditions imposed in § 3, we have the further constraint that the horizontal flux is trapped between the planes $z = 0, 1$:

$$\int_0^1 \mathbf{H}_0(z)\,dz = (1, 0, 0), \tag{A 3}$$

with the normalization adopted above. It is evident from symmetry considerations that

$$\mathbf{H}_0(z) = (H_0(z), 0, 0) \tag{A 4}$$

† The methods used are essentially the methods of 'mean field electrodynamics' as applied to space-periodic velocity fields by Childress (1970) and Roberts (1970).

and so (A 2) becomes

$$dH_0/dz = \epsilon \mathscr{E}_y, \tag{A 5}$$

the constant of integration being zero, since $\mathscr{E}_y = 0$ and $dH_0/dz = 0$ on the boundaries $z = 0, 1$ [equations (11) and (12)].

When $\epsilon \ll 1$, we expand $\mathbf{H}$ as a power series

$$\mathbf{H}(\mathbf{x}) = \sum_{n=0}^{\infty} \epsilon^n \mathbf{H}^{(n)}(\mathbf{x}), \tag{A 6}$$

where

$$-\nabla^2 . \mathbf{H}^{(n+1)} = \nabla \wedge (\mathbf{u} \wedge \mathbf{H}^{(n)}), \tag{A 7}$$

and $\mathbf{H}^{(0)} = (1, 0, 0)$. When $\mathbf{u}(\mathbf{x})$ is periodic, these equations may be solved successively by elementary methods, making repeated use of the identity

$$-\nabla^{-2} \cos(lx+p)\cos(my+q)\cos(nz+r) = (l^2+m^2+n^2)^{-1}\cos(lx+p)\cos(my+q)\cos(nz+r). \tag{A 8}$$

If $\mathscr{E}^{(n)} = \langle \mathbf{u} \wedge \mathbf{H}^{(n)} \rangle$ and $\mathbf{H}_0^{(n)} = \langle \mathbf{H}^{(n)} \rangle$, then (A 5) gives

$$dH_0^{(n+1)}/dz = \mathscr{E}_y^{(n)}. \tag{A 9}$$

Clearly $\mathscr{E}_y^{(0)} = 0$ and $H_0^{(1)} = 0$. The constraint (A 3) implies that

$$\int_0^1 H_0^{(n)}(z)\, dz = 0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{A 10}$$

With $n = 0$, (A 7) gives

$$-\nabla^2 \mathbf{H}^{(1)} = \nabla \wedge (\mathbf{u} \wedge \mathbf{H}^{(0)}) = \mathbf{H}^{(0)}.\nabla \mathbf{u} = \partial \mathbf{u}/\partial x, \tag{A 11}$$

and when $\mathbf{u}$ is given by† [equation (3)]

$$\mathbf{u} = (-\sin x(1+\tfrac{1}{2}\cos y)\cos z, \; -(1+\tfrac{1}{2}\cos x)\sin y \cos z, \; (\cos x + \cos y + \cos x \cos y)\sin z) \tag{A 12}$$

it follows that

$$\mathbf{H}^{(1)} = \tfrac{1}{6}(-\cos x(3+\cos y)\cos z, \; \sin x \sin y \cos z, \; -(3+2\cos y)\sin x \sin z). \tag{A 13}$$

Hence

$$\mathscr{E}_y^{(1)} = \langle u_z H_x^{(1)} - u_x H_z^{(1)} \rangle = -\tfrac{7}{24}\sin 2z, \tag{A 14}$$

and, from (A 9) and (A 10),

$$H_0^{(2)} = \tfrac{7}{48}\cos 2z. \tag{A 15}$$

Hence, to order ϵ^2, the mean field is symmetrically perturbed about the centre-plane $z = \frac{1}{2}\pi$; it is slightly intensified for $z < \frac{1}{4}\pi$ *and* for $z > \frac{3}{4}\pi$ owing to the horizontal divergence of the flow in these regions. In order to obtain the pumping (or valve) effect discovered (see § 2), it is necessary to continue to the next order in ϵ.

For this purpose, we need only calculate

$$\mathscr{E}_y^{(2)} = \langle u_z H_x^{(2)} - u_x H_z^{(2)} \rangle, \tag{A 16}$$

† We put $\mathbf{x}^* = \pi \mathbf{x}$ and drop the asterisk for convenience of notation; the magnetic Reynolds number used in §§ 3 and 4 above is then $R_m = \pi\epsilon$.

and we need therefore retain only those terms of the horizontal Fourier expansion of $\mathbf{H}^{(2)}$ involving $\sin x$, $\cos x$, $\sin y$ and $\cos y$. For example, we have

$$
\begin{aligned}
(\mathbf{u} \wedge \mathbf{H}^{(1)})_x &= u_y H_z^{(1)} - u_z H_y^{(1)} \\
&= \tfrac{1}{48}(12 \sin x \sin y + \sin y \sin 2x + 2 \sin x \sin 2y) \sin 2z, \qquad \text{(A 17)}
\end{aligned}
$$

and we need only retain

$$(\mathbf{u} \wedge \mathbf{H}^{(1)})_x = \tfrac{1}{4} \sin x \sin y \sin 2z + \ldots, \qquad \text{(A 18)}$$

in calculating $\mathbf{H}^{(2)}$ from (A 7). Similarly, we have

$$(\mathbf{u} \wedge \mathbf{H}^{(1)})_y = -\tfrac{1}{48} \sin 2z(14 + 2\cos x + 15 \cos y + 12 \cos x \cos y + \ldots), \qquad \text{(A 19)}$$

and

$$(\mathbf{u} \wedge \mathbf{H}^{(1)})_z = -\tfrac{1}{48}(1 + \cos 2z)(5 \sin y + 12 \cos x \sin y + \ldots). \qquad \text{(A 20)}$$

We can immediately calculate the components of $\nabla \wedge (\mathbf{u} \wedge \mathbf{H}^{(1)})$, and from (A 7), using (A 8), we then have

$$
\begin{aligned}
H_x^{(2)} &= -\tfrac{1}{48} \cos y(5 + 6\cos x) \\
&\quad + \tfrac{1}{240} \cos 2z(35 + 4\cos x + 25 \cos y + 10 \cos x \cos y + \ldots), \qquad \text{(A 21)}
\end{aligned}
$$

and

$$H_z^{(2)} = \tfrac{1}{120} \sin 2z(\sin x + \ldots). \qquad \text{(A 22)}$$

Hence from (A 16),

$$\mathscr{E}_y^{(2)} = -\tfrac{37}{240} \sin z + \tfrac{3}{40} \cos z \sin 2z, \qquad \text{(A 23)}$$

and so, from (A 9) and (A 10),

$$H_0^{(3)}(z) = \tfrac{1}{240} \cos z(37 - 12\cos^2 z) = \tfrac{7}{60} \cos z - \tfrac{1}{80} \cos 3z. \qquad \text{(A 24)}$$

This contribution to the field is antisymmetric about $z = \frac{1}{2}\pi$; in fact the flux of $H_0^{(3)}$ in the lower half $0 < z < \frac{1}{2}\pi$ is

$$\Phi^{(3)} = \int_0^{\frac{1}{2}\pi} H_0^{(3)}(z)\, dz = \tfrac{29}{240} \doteqdot 0{\cdot}121, \qquad \text{(A 25)}$$

and the flux in the upper half $\frac{1}{2}\pi < z < \pi$ is $-0{\cdot}121$. Continuation of the process shows clearly that $H_0^{(n)}(z)$ is symmetric or antisymmetric about $z = \frac{1}{2}\pi$ according as n is even or odd. The mean field $H_0(z)$ is now given to order ϵ^3 by

$$H_0(z) = 1 + \frac{7\epsilon^2}{48} \cos 2z + \frac{\epsilon^3}{240}(28 \cos z - 3 \cos 3z) + O(\epsilon^4). \qquad \text{(A 26)}$$

It seems likely that the first three terms give a reasonable approximation for $\epsilon \lesssim 1$.

Comparison of (A 26) with (15) shows that

$$H_{001}^x = \frac{7\epsilon^3}{60} + O(\epsilon^5), \quad H_{002}^x = \frac{7\epsilon^2}{48} + O(\epsilon^4), \quad H_{003}^x = \frac{-\epsilon^3}{80} + O(\epsilon^5), \qquad \text{(A 27)}$$

results that are not inconsistent with figure 3 (with $\epsilon = R_m/\pi$).

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